ANOTHER PROOF OF SCURRY'S CHARACTERIZATION OF A TWO WEIGHT NORM INEQUALITY FOR A SEQUENCE-VALUED POSITIVE DYADIC OPERATOR

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Abstract. In this note we prove Scurry's testing conditions for the boundedness of a sequence-valued averaging positive dyadic operator from a weighted Lp space to a sequence-valued weighted Lp space by using parallel stopping cubes.

Contents

1. Introduction and statement of the theorem

Let λ_Q be non-negative real numbers indexed by the dyadic cubes $Q \in \mathcal{D}$ of $\mathbb{R}^d$. We define the operator T by

$$T(f) := (\lambda_Q \langle f \rangle_Q 1_Q)_{Q \in \mathcal{D}}.$$

Suppose that $1 < p < \infty$. Let u and ω be weights. We are considering sufficient and necessary testing conditions for the boundedness of the operator $T : L^p(u) \to L^p_{\ell^r}(\omega)$. By the change of weight $\sigma = u^{-1/(p-1)}$ we may as well study the boundedness of the operator $T(\,\cdot\,\sigma) : L^p(\sigma) \to L^p_{\ell^r}(\omega)$.

In the case $r = \infty$ Sawyer [5, Theorem A] proved that for $\lambda_Q = |Q|^{a/d}$ with $0 \leq a < d$ it is sufficient to test the boundedness of the operator $T(\,\cdot\,\sigma) : L^p(\sigma) \to L^p_{\ell^\infty}(\omega)$ on functions $f = 1_R$ with $R \in \mathcal{D}$. This testing condition holds for every λ_Q, as

Date: April 30, 2013.
2010 *Mathematics Subject Classification.* 42B20,46E40.
Key words and phrases. weight,vector-valued,testing condition.
The author is supported by the European Union through the ERC Starting Grant "Analytic-probabilistic methods for borderline singular integrals".

one can check by using the well-known proof in which one linearizes the operator $|Tf|_\infty = \sum_{Q\in\mathcal{D}} \lambda_Q \langle f\rangle_Q 1_{E(Q)}$ by using the partition

$$E(Q) := \{x \in Q : |Tf(x)|_\infty = \lambda_Q\langle f\rangle_Q \text{ and } \lambda_{Q'}\langle f\rangle_{Q'} < \lambda_Q\langle f\rangle_Q \text{ whenever } Q' \supsetneq Q\}$$

and applies the dyadic Carleson embedding theorem. The exact statement of the testing condition in the case $r = \infty$ corresponds to Theorem 1.1 with $r = \infty$ and with the dual testing (1.2b) omitted.

In the case $r = 1$ the boundedness of the sequence-valued operator $T(\cdot\sigma) : L^p(\sigma) \to L^p_{\ell^1}(\omega)$ is equivalent to the boundedness of the real-valued operator $S(\cdot\sigma) : L^p(\sigma) \to L^p(\omega)$ defined by

$$Sf := |Tf|_1 = \sum_{Q\in\mathcal{D}} \lambda_Q\langle f\rangle_Q 1_Q.$$

For the boundedness of $S(\cdot\sigma) : L^p(\sigma) \to L^p(\omega)$ it is sufficient to test the boundedness of both the operator $S(\cdot\sigma) : L^p(\sigma) \to L^p(\omega)$ and its formal adjoint $S(\cdot\omega) : L^{p'}(\omega) \to L^{p'}(\sigma)$ on functions $f = 1_R$ with $R \in \mathcal{D}$. These testing conditions were proven for $p = 2$

- by Nazarov, Treil, and Volberg [4] by the Bellman function technique

and for $1 < p < \infty$

- by Lacey, Sawyer, and Uriarte-Tuero [3] by techniques that are similar to the ones that Sawyer [6] used in proving such testing conditions for a large class of integral operators $I(\cdot\sigma) : L^p(\sigma) \to L^p(\omega)$ with non-negative kernels (in particular for fractional integrals and Poisson integrals),
- by Treil [8] by splitting the summation over dyadic cubes $Q \in \mathcal{D}$ in the dual pairing $\langle Sf, g\rangle_{L^p(\omega)\times L^{p'}(\omega)}$ by the condition "$\sigma(Q)(\langle f\rangle^\sigma_Q)^p > \omega(Q)(\langle g\rangle^\omega_Q)^{p'}$",
- and by Hytönen [1, Section 6] by constructing stopping cubes for each of the pairs (f, σ) and (g, ω) in parallel and then splitting the summation in the dual pairing $\langle Sf, g\rangle_{L^p(\omega)\times L^{p'}(\omega)}$ by the condition "$\pi_{\mathcal{F}}(Q) \subseteq \pi_{\mathcal{G}}(Q)$". The technique of organizing the summation by parallel stopping cubes is from the work of Lacey, Sawyer, Shen and Uriarte-Tuero [2] on the two-weight boundedness of the Hilbert transform.

The exact statement of the testing conditions for the operator $S(\cdot\sigma) : L^p(\sigma) \to L^p(\omega)$ corresponds to Theorem 1.2, which is equivalent to Theorem 1.1 with $r = 1$ for the operator $T(\cdot\sigma) : L^p(\sigma) \to L^p_{\ell^1}(\omega)$, as explained in Remark 1.3.

In the case $1 < r < \infty$ the testing conditions in Theorem 1.1 for the boundedness of the operator $T(\cdot\sigma) : L^p(\sigma) \to L^p_{\ell^r}(\omega)$ were first proven by Scurry [7] by adapting Lacey, Sawyer, and Uriarte-Tuero's proof of the case $r = 1$ to the case $1 < r < \infty$. In this note we adapt Hytönen's proof of the case $r = 1$ to the case $1 < r < \infty$.

Next we state Theorem 1.1. Note that the formal adjoint $T^* : L^{p'}_{\ell^{r'}} \to L^{p'}$ of the operator $T : L^p \to L^p_{\ell^r}$ is given by

$$T^*(g) = \sum_{Q\in\mathcal{D}} \lambda_Q\langle g_Q\rangle_Q 1_Q.$$

The operator T is positive in the sense that if $f \geq 0$, then $(Tf)_Q \geq 0$ for every $Q \in \mathcal{D}$. Likewise, the operator T^* is positive in the sense that if $g_Q \geq 0$ for every $Q \in \mathcal{D}$, then $T^*(g) \geq 0$. For each dyadic cube R we define the localized version T_R of the operator T by

$$T_R(f) := (\lambda_Q\langle f\rangle_Q 1_Q)_{\substack{Q\in\mathcal{D}:\\ Q\subseteq R}}.$$

Hence the formal adjoint $T_R^*: L^{p'}_{\ell^{r'}} \to L^{p'}$ of the operator $T_R: L^p \to L^p_{\ell^r}$ is given by

$$T_R^*(g) = \sum_{\substack{Q\in\mathcal{D}:\\ Q\subseteq R}} \lambda_Q \langle g_Q\rangle_Q 1_Q.$$

Note that for the formal adjoint $(T(\cdot\sigma))^* : L^{p'}_{\ell^{r'}}(\omega) \to L^{p'}(\sigma)$ of the operator $T(\cdot\sigma): L^p(\sigma) \to L^p_{\ell^r}(\omega)$ we have $(T(\cdot\sigma))^* = T^*(\cdot\omega)$.

Theorem 1.1. *Let $1 \le r \le \infty$ and $1 < p < \infty$. Suppose that σ and ω are locally integrable positive functions. Then the two weight norm inequality*

$$\|T(f\sigma)\|_{L^p_{\ell^r}(\omega)} \le \tilde{C}\|f\|_{L^p(\sigma)} \tag{1.1}$$

holds if and only if both of the following testing conditions hold

$$\|T_R(\sigma)\|_{L^p_{\ell^r}(\omega)} \le C\sigma(R)^{1/p} \quad \textit{for all } R\in\mathcal{D} \tag{1.2a}$$

$$\|T_R^*(g\omega)\|_{L^{p'}(\sigma)} \le C^*\|g\|_{L^\infty_{\ell^{r'}}(\omega)}\omega(R)^{1/p'} \quad \textit{for all } g = (a_Q 1_Q)_{Q\in\mathcal{D}} \textit{ with constants } a_Q \ge 0. \tag{1.2b}$$

Moreover, $\tilde{C} \le C_{p',r'} 20pp'(C+C^*)$, where $C_{p',r'}$ is the constant of Stein's inequality.

Remark 1.1 (Restrictions on the test functions in the dual testing condition). The dual testing condition (1.2b) for all functions g is equivalent to the dual testing condition restricted to functions g such that $|g(x)|_{r'} = 1$ for ω-almost every $x\in\mathbb{R}^d$, which is seen as follows. Suppose that $\|g\|_{L^\infty_{\ell^{r'}}(\omega)} < \infty$. Then $|g_Q| \le \|g\|_{L^\infty_{\ell^{r'}}(\omega)} \frac{1}{|g|_{r'}}|g_Q|$ for every $Q\in\mathcal{D}$ ω-almost everywhere. Note that $|\frac{1}{|g|_{r'}}(|g_Q|)_{Q\in\mathcal{D}}|_{r'} = 1$ ω-almost everywhere. By the positivity and the linearity of the operator T^* we have

$$|T^*((g_Q)_{Q\in\mathcal{D}}\omega)| \le T^*((|g_Q|)_{Q\in\mathcal{D}}\omega) \le \|g\|_{L^\infty_{\ell^{r'}}(\omega)} T^*(\frac{1}{|g|_{r'}}(|g_Q|)_{Q\in\mathcal{D}}\omega).$$

Moreover, the dual testing condition (1.2b) for all functions $g = (g_Q)_{Q\in\mathcal{D}}$ is equivalent to the dual testing condition restricted to piecewise constant functions $g = (a_Q 1_Q)_{Q\in\mathcal{D}}$, as observed in Section 2.2.

Remark 1.2 (Sufficient condition for the dual condition). The condition

$$\|T_R(\omega)\|_{L^{p'}_{\ell^r}(\sigma)} \le C^*\omega(R)^{1/p'} \quad \text{for all } R\in\mathcal{D} \tag{1.3}$$

implies the dual testing condition (1.2b). This is seen as follows. We have that

$$\begin{aligned}
&T_R^*((a_Q 1_Q)_{Q\in\mathcal{D}}\,\omega)\\
&= \sum_{\substack{Q\in\mathcal{D}:\\ Q\subseteq R}} \lambda_Q a_Q 1_Q \langle\omega\rangle_Q && \text{by the definition of } T_R^*\\
&\le |(a_Q 1_Q)_{Q\in\mathcal{D}}|_{r'} |(\lambda_Q\langle\omega\rangle_Q 1_Q)_{\substack{Q\in\mathcal{D}:\\ Q\subseteq R}}|_r && \text{by Hölder's inequality}\\
&\le \|(a_Q 1_Q)\|_{L^\infty_{\ell^{r'}}} |T_R(\omega)|_r && \text{by the definition of } T_R.
\end{aligned}$$

Hence by (1.3) we have

$$\begin{aligned}
\|T_R^*((a_Q 1_Q)_{Q\in\mathcal{D}}\,\omega)\|_{L^{p'}(\sigma)} &\le \|(a_Q 1_Q)\|_{L^\infty_{\ell^{r'}}} \|T_R(\omega)\|_{L^{p'}_{\ell^r}(\sigma)}\\
&\le C^*\|(a_Q 1_Q)\|_{L^\infty_{\ell^{r'}}(\omega)}\omega(R)^{1/p'}.
\end{aligned}$$

Remark 1.3 (In the case $r = 1$ we may consider a real-valued operator). Consider the real-valued operator S defined by

$$Sf := |Tf|_1 = \sum_{Q\in\mathcal{D}} \lambda_Q \langle f\rangle_Q 1_Q.$$

Note that in this notation the direct testing condition (1.2a) is written as

$$\|S_R(\sigma)\|_{L^p(\omega)} \le C\sigma(R)^{1/p}.$$

Observe that the operator $S : L^p \to L^p$ is formally self-adjoint and that for the adjoint $(S(\cdot\sigma))^* : L^{p'}(\omega) \to L^{p'}(\sigma)$ of the operator $S(\cdot\sigma) : L^p(\sigma) \to L^p(\omega)$ we have $S(\cdot\sigma))^* = S(\cdot\omega)$. By Remark 1.2 the dual testing condition (1.2b) is implied by the dual testing condition

$$\|S_R(\omega)\|_{L^{p'}(\sigma)} \le C^*\omega(R)^{1/p'}, \tag{1.4}$$

and, conversely, the dual testing condition (1.4) is implied by the dual testing condition (1.2b) applied to the function $g = (1_Q)_{Q\in\mathcal{D}}$. Therefore Theorem 1.1 in the case $r = 1$ is equivalent to the following theorem.

Theorem 1.2. *Let $1 < p < \infty$. Suppose that σ and ω are locally integrable positive functions. Then the two weight norm inequality*

$$\|S(f\sigma)\|_{L^p(\omega)} \le \tilde{C}\|f\|_{L^p(\sigma)} \tag{1.5}$$

holds if and only if both of the following testing conditions hold

$$\|S_R(\sigma)\|_{L^p(\omega)} \le C\sigma(R)^{1/p} \quad \textit{for all } R\in\mathcal{D} \tag{1.6a}$$

$$\|S_R(\omega)\|_{L^{p'}(\sigma)} \le C\omega(R)^{1/p'} \quad \textit{for all } R\in\mathcal{D}. \tag{1.6b}$$

2. Proof of the theorem in the case $1 \le r \le \infty$

Notation. We use the following standard notation: $\langle f\rangle_Q^\sigma := \frac{1}{\sigma(Q)}\int_Q f\sigma$, $\langle f\rangle_Q := \frac{1}{|Q|}\int_Q f$, and $|g|_{r'} := \|g\|_{\ell^{r'}}$.

Proof of the necessity of the testing conditions. By duality the norm inequality (1.1) for the operator $T(\cdot\sigma) : L^p(\sigma) \to L^p_{\ell^r}(\omega)$ is equivalent to the norm inequality

$$\|T^*(g\omega)\|_{L^{p'}(\sigma)} \le \tilde{C}\|g\|_{L^{p'}_{\ell^{r'}}(\omega)} \tag{2.1}$$

for the adjoint operator $T^*(\cdot\omega) : L^{p'}_{\ell^{r'}}(\omega) \to L^{p'}(\sigma)$. The necessity of the direct testing condition (1.2a) follows by applying the norm inequality (1.1) for functions $f = 1_R$ and the necessity of the dual testing condition (1.2b) follows by applying the norm inequality (2.1) for functions $g1_R$ and using the estimate

$$\|g1_R\|_{L^{p'}_{\ell^{r'}}(\omega)} \le \|g\|_{L^\infty_{\ell^{r'}}(\omega)}\omega(R)^{1/p'}.$$

□

Proof of the sufficiency of the testing conditions. By duality the norm inequality (1.1) is equivalent the following norm inequality for the dual pairing

$$\langle T(f\sigma), g\rangle_{L^p_{\ell^r}(\omega)\times L^{p'}_{\ell^{r'}}(\omega)} \le \tilde{C}\|f\|_{L^p(\sigma)}\|g\|_{L^{p'}_{\ell^{r'}}(\omega)}.$$

2.1. **Reductions.**

Claim (Reduction). We may assume that $f \geq 0$, $g_Q \geq 0$ for every $Q \in \mathcal{D}$, and $g = (a_Q 1_Q)_{Q \in \mathcal{D}}$ for some constants $a_Q \geq 0$. Moreover, we may assume that the collection $\mathcal{D}$ is finite and that for some $Q_0 \in \mathcal{D}$ we have $Q \subseteq Q_0$ for all $Q \in \mathcal{D}$.

Proof of the claim. Since

$$|\langle T(f\sigma), (g_Q)_{Q\in\mathcal{D}}\rangle_{L^p_{\ell^r}(\omega)\times L^{p'}_{\ell^{r'}}(\omega)}| \leq \langle T(|f|\sigma), (|g_Q|)_{Q\in\mathcal{D}}\rangle_{L^p_{\ell^r}(\omega)\times L^{p'}_{\ell^{r'}}(\omega)},$$

$\|f\|_{L^p(\sigma)} = \||f|\|_{L^p(\sigma)}$ and $\|(g_Q)_{Q\in\mathcal{D}}\|_{L^{p'}_{\ell^{r'}}(\omega)} = \|(|g_Q|)_{Q\in\mathcal{D}}\|_{L^{p'}_{\ell^{r'}}(\omega)}$, we may assume that $g_Q \geq 0$ and $f \geq 0$. By the monotone convergence theorem we may assume that the collection $\mathcal{D}$ is finite and that all dyadic cubes in the collection $\mathcal{D}$ are contained in some dyadic cube Q_0. We observe that

$$\begin{aligned} T^*((g_Q)\omega) &= \sum_{Q\in\mathcal{D}} \lambda_Q \langle g_Q \omega\rangle_Q 1_Q = \sum_{Q\in\mathcal{D}} \lambda_Q \langle g_Q\rangle^\omega_Q \langle \omega\rangle_Q 1_Q \\ &= \sum_{Q\in\mathcal{D}} \lambda_Q \langle \langle g_Q\rangle^\omega_Q 1_Q \omega\rangle_Q 1_Q = T^*((\langle g_Q\rangle^\omega_Q 1_Q)\omega). \end{aligned} \tag{2.2}$$

For $1 \leq r' \leq \infty$ and $1 < p' < \infty$ we have by Stein's inequality

$$\|(\langle g_Q\rangle^\omega_Q 1_Q)\|_{L^{p'}_{\ell^{r'}}(\omega)} \leq C_{p',r'} \|(g_Q 1_Q)\|_{L^{p'}_{\ell^{r'}}(\omega)}$$

for

$$C_{p',r'} = \begin{cases} (\frac{p'}{r'})^{1/r'} & \text{if } p' \geq r' \\ (\frac{p}{r})^{1/r} & \text{if } p' < r' \end{cases}. \tag{2.3}$$

Hence we may assume that the function g is piecewise constant in the sense that $g = (a_Q 1_Q)$ for some constants $a_Q \geq 0$. □

Remark 2.1. The constant (2.3) in Stein's inequality can be checked in the following well-known way. Let $(\mathcal{F}_k)_{k\in\mathbb{Z}}$ be a filtration. By Doob's inequality

$$\|(\mathbb{E}(f|\mathcal{F}_k))_{k\in\mathbb{Z}}\|_{L^p_{\ell^\infty}} \leq p' \|f\|_{L^p}$$

for all $1 < p \leq \infty$ and for all nonnegative functions f. From this it follows directly that

$$\|(\mathbb{E}(g_k|\mathcal{F}_k)_{k\in\mathbb{Z}}\|_{L^p_{\ell^\infty}} \leq \|(\mathbb{E}(|g_k|_\infty|\mathcal{F}_k)_{k\in\mathbb{Z}}\|_{L^p_{\ell^\infty}} \leq p' \|(g_k)_{k\in\mathbb{Z}}\|_{L^p_{\ell^\infty}}$$

and by using duality that

$$\|(\mathbb{E}(g_k|\mathcal{F}_k))_{k\in\mathbb{Z}}\|_{L^p_{\ell^1}} \leq p \|(g_k)_{k\in\mathbb{Z}}\|_{L^p_{\ell^1}} \tag{2.4}$$

for all $1 \leq p < \infty$ and for all nonnegative functions $(g_k)_{k\in\mathbb{Z}}$. Hence in the case $p/r \geq 1$ we have by Jensen's inequality and by the inequality (2.4) that

$$\begin{aligned} &\|(\mathbb{E}(g_k|\mathcal{F}_k))_{k\in\mathbb{Z}}\|_{L^p_{\ell^r}} = \|(\mathbb{E}(g_k|\mathcal{F}_k)^r)_{k\in\mathbb{Z}}\|^{1/r}_{L^{p/r}_{\ell^1}} \\ &\leq \|(\mathbb{E}(g_k^r|\mathcal{F}_k))_{k\in\mathbb{Z}}\|^{1/r}_{L^{p/r}_{\ell^1}} \leq (\frac{p}{r})^{1/r} \|(g_k^r)_{k\in\mathbb{Z}}\|^{1/r}_{L^{p/r}_{\ell^1}} = (\frac{p}{r})^{1/r} \|(g_k)_{k\in\mathbb{Z}}\|_{L^p_{\ell^r}}. \end{aligned}$$

Case $p/r < 1$ can be checked by using duality.

2.2. **Constructing stopping cubes and organizing the summation.** Next we define recursively stopping cubes for each of the pairs (f,σ) and (g,ω).

Claim (Construction and properties of the stopping cubes related to the pair (g,ω)). Let $\mathrm{ch}_{\mathcal{G}}(G)$ be the collection of all the maximal dyadic subcubes G' of G such that

$$|(a_Q)_{\substack{Q\in\mathcal{D}:\\ Q\supseteq G'}}|_{r'} > 2\langle|g|_{r'}\rangle^{\omega}_{G}\ . \tag{2.5}$$

Define recursively $\mathcal{G}_0 := \{Q_0\}$ and $\mathcal{G}_{k+1} := \bigcup_{G\in\mathcal{G}_k}\mathrm{ch}_{\mathcal{G}}(G)$. Let $\mathcal{G} := \bigcup_{k=0}^{\infty}\mathcal{G}_k$. Let

$$E_{\mathcal{G}}(G) := G \setminus \bigcup_{G'\in\mathrm{ch}_{\mathcal{G}}(G)} G'.$$

Define $\pi_{\mathcal{G}}(Q)$ as the minimal $G\in\mathcal{G}$ such that $Q\subseteq G$. Then the following properties hold:

- (b1) The sets $\{G'\}_{G'\in\mathrm{ch}_{\mathcal{G}}(G)}\cup\{E_{\mathcal{G}}(G)\}$ partition G.
- (b2) The collection $\{E_{\mathcal{G}}(G)\}_{G\in\mathcal{G}}$ is pairwise disjoint.
- (b3) $\omega(E_{\mathcal{G}}(G)) \geq \frac{1}{2}\omega(G)$.
- (b4) $|(a_Q)_{\substack{Q\in\mathcal{D}:\\ Q\supseteq R}}|_{r'} \leq 2\langle|g|_{r'}\rangle^{\omega}_{\pi_{\mathcal{G}}(R)}$
- (b5) $\|(g_Q)_{\substack{Q\in\mathcal{D}:\\ \pi_{\mathcal{G}}(Q)=G}}\|_{L^{\infty}_{\ell^{r'}}(\omega)} \leq 2\langle|g|_{r'}\rangle^{\omega}_{G}\ .$

Proof of the claim. The property (b1) holds because $G'\in\mathrm{ch}_{\mathcal{G}}(G)$ are maximal subcubes of G and $E_{\mathcal{G}}(G)$ is the complement of $\bigcup_{G'\in\mathrm{ch}_{\mathcal{G}}(G)}G'$ in G. Next we check the property (b2). By definition of the set $E_{\mathcal{G}}(G)$ we have that the collection $\{E_{\mathcal{G}}(G)\}\cup\{G'\}_{G'\in\mathrm{ch}_{\mathcal{G}}(G)}$ is pairwise disjoint. Since $E_{\mathcal{G}}(G')\subseteq G'$, the collection $\{E_{\mathcal{G}}(G)\}\cup\{E_{\mathcal{G}}(G')\}_{G'\in\mathrm{ch}_{\mathcal{G}}(G)}$ is pairwise disjoint. This together with the recursive definition of the collection $\mathcal{G}$ implies by induction that the collection $\{E_{\mathcal{G}}(G)\}_{G\in\mathcal{G}}$ is pairwise disjoint.

Next we prove the property (b3). We have

$$\begin{aligned}
\langle|g|_{r'}\rangle^{\omega}_{G} &= \sum_{G'\in\mathrm{ch}_{\mathcal{G}}(G)}\langle|g|_{r'}\rangle^{\omega}_{G'}\frac{\omega(G')}{\omega(G)} + \langle|g|_{r'}\rangle^{\omega}_{E_{\mathcal{G}}(G)}\frac{\omega(E_{\mathcal{G}}(G))}{\omega(G)} && \text{the law of total expectation}\\
&\geq \sum_{G'\in\mathrm{ch}_{\mathcal{G}}(G)}\langle|g|_{r'}\rangle^{\omega}_{G'}\frac{\omega(G')}{\omega(G)}\\
&\geq \sum_{G'\in\mathrm{ch}_{\mathcal{G}}(G)}|(a_Q)_{\substack{Q\in\mathcal{D}:\\ Q\supseteq G'}}|_{r'}\frac{\omega(G')}{\omega(G)}\\
&\geq 2\langle|g|_{r'}\rangle^{\omega}_{G}\sum_{G'\in\mathrm{ch}_{\mathcal{G}}(G)}\frac{\omega(G')}{\omega(G)} && \text{by (2.5).}
\end{aligned}$$

Hence

$$\sum_{G'\in\mathrm{ch}_{\mathcal{G}}(G)}\omega(G') \leq \frac{1}{2}\omega(G),$$

which by the definition $E_{\mathcal{G}}(G) := G\setminus\bigcup_{G'\in\mathrm{ch}_{\mathcal{G}}(G)}G'$ is equivalent to

$$\omega(E_{\mathcal{G}}(G)) \geq \frac{1}{2}\omega(G).$$

Next we prove (b4). Assume that $\pi_{\mathcal{G}}(R) = G$. By definition this means that $G\in\mathcal{G}$ is such that $R\subseteq G$ and that there is no $G'\in\mathcal{G}$ such that $R\subseteq G'\subsetneq G$. If we had

$$|(a_Q)_{\substack{Q\in\mathcal{D}:\\ Q\supseteq R}}|_{r'} > 2\langle|g|_{r'}\rangle^{\omega}_{G},$$

then by definition of the collection $\mathrm{ch}_{\mathcal{G}}(G)$ there would be $G' \in \mathrm{ch}_{\mathcal{G}}(G)$ such that $R \subseteq G'$ and $G' \subsetneq G$, in which case $R \subseteq G' \subsetneq G$. Therefore by contrapositive we have

$$|(a_Q)_{\substack{Q\in\mathcal{D}:\\ Q\supseteq R}}|_{r'} \leq 2\langle |g|_{r'}\rangle^{\omega}_{G}.$$

Next we prove (b5). Observe that the function $x \mapsto (a_Q 1_Q(x))_{\substack{Q\in\mathcal{D}:\\ \pi_{\mathcal{G}}(Q)=G}}$ is supported on $\bigcup_{Q\in\mathcal{D}:\pi_{\mathcal{G}}(Q)=G} Q$. Let x be in the support of the function. Let Q_x be the minimal $Q \in \mathcal{D}$ such that $Q \ni x$ and $\pi_{\mathcal{G}}(Q) = G$. By the piecewise constancy and the property (b4) we have

$$|(a_Q 1_Q(x))_{\substack{Q\in\mathcal{D}:\\ \pi_{\mathcal{G}}(Q)=G}}|_{r'} = |(a_Q)_{\substack{Q\in\mathcal{D}:\\ \pi_{\mathcal{G}}(Q)=G \text{ and } Q\supseteq Q_x}}|_{r'} \leq |(a_Q)_{\substack{Q\in\mathcal{D}:\\ Q\supseteq Q_x}}|_{r'} \leq 2\langle |g|_{r'}\rangle^{\omega}_{\pi_{\mathcal{G}}(Q_x)} = 2\langle |g|_{r'}\rangle^{\omega}_{G}.$$

This completes the proof of the claim. □

For the pair (f,σ) we choose the stopping cubes as in the case $r = 1$, which is as follows. Let $\mathrm{ch}_{\mathcal{F}}(F)$ be the collection of all maximal dyadic subcubes F' of F such that

$$\langle f\rangle^{\sigma}_{F'} > 2\langle f\rangle^{\sigma}_{F}. \tag{2.6}$$

Define recursively $\mathcal{F}_0 \coloneqq \{Q_0\}$ and $\mathcal{F}_{k+1} \coloneqq \bigcup_{F\in\mathcal{F}_k} \mathrm{ch}_{\mathcal{F}}(F)$. Let $\mathcal{F} \coloneqq \bigcup_{k=0}^{\infty} \mathcal{F}_k$. Let

$$E_{\mathcal{F}}(F) \coloneqq F \setminus \bigcup_{F'\in\mathrm{ch}_{\mathcal{F}}(F)} F'.$$

Define $\pi_{\mathcal{F}}(Q)$ as the minimal $F \in \mathcal{F}$ such that $Q \subseteq F$. The construction has the following well-known properties.

(a1) The sets $F' \in \mathrm{ch}_{\mathcal{F}}(F)$ and $E_{\mathcal{F}}(F)$ partition F.
(a2) The collection $\{E_{\mathcal{F}}(G)\}_{G\in\mathcal{F}}$ is pairwise disjoint.
(a3) $\sigma(E_{\mathcal{F}}(F)) \geq \frac{1}{2}\sigma(F)$.
(a4) $\langle f\rangle^{\sigma}_{Q} \leq 2\langle f\rangle^{\sigma}_{\pi_{\mathcal{F}}(Q)}$.

Next we split the summation in the dual pairing by using the stopping cubes. Let $\pi(Q) = (F,G)$ denote that $\pi_{\mathcal{F}}(Q) = F$ and $\pi_{\mathcal{G}}(Q) = G$.

$$\langle T(f\sigma), g\rangle_{L^p_{\ell^r}(\omega)\times L^{p'}_{\ell^{r'}}(\omega)} = \sum_{Q\in\mathcal{D}} \lambda_Q \langle f\rangle^{\sigma}_{Q}\langle\sigma\rangle_Q \langle g_Q\rangle^{\omega}_{Q}\langle\omega\rangle_Q |Q|$$

$$\leq \sum_{G\in\mathcal{G}} \Big(\sum_{\substack{F\in\mathcal{F}\\ F\subseteq G}} \sum_{\substack{Q\in\mathcal{D}\\ \pi(Q)=(F,G)}} \lambda_Q \langle g_Q\omega\rangle_Q \int_Q f\sigma \Big) \tag{2.7a}$$

$$+ \sum_{F\in\mathcal{F}} \Big(\sum_{\substack{G\in\mathcal{G}\\ G\subseteq F}} \sum_{\substack{Q\in\mathcal{D}\\ \pi(Q)=(F,G)}} \lambda_Q \langle f\rangle^{\sigma}_{Q}\langle\sigma\rangle_Q \int_Q g_Q\omega \Big). \tag{2.7b}$$

Remark 2.2. As explained in Remark 1.3, in the case $r = 1$ we may deal symmetrically with the pairs (f,σ) and (g,ω). Hence in the case $r = 1$ we may impose the stopping condition

$$\langle g\rangle^{\omega}_{G'} > 2\langle g\rangle^{\omega}_{G}.$$

for the real-valued function g, as it is done in Hytönen's proof [1, Section 6] of the case $r = 1$, whereas in the case $1 < r < \infty$ we reduce the sequence-valued function $g = (g_Q)$ to the piecewise constant function $g = (a_Q 1_Q)$ and impose the stopping condition

$$|(a_Q)_{\substack{Q\in\mathcal{D}:\\ Q\supseteq G'}}|_{r'} > 2\langle |g|_{r'}\rangle^{\omega}_{G}\ .$$

2.3. **Lemma.** The following well-known lemma will be used in Section 2.4 and in Section 2.5.

Lemma 2.1 (Special case of dyadic Carleson embedding theorem)**.** *Let $1 < p < \infty$. Suppose that $\{E_{\mathcal{F}}(F)\}_{F\in\mathcal{F}}$ is a pairwise disjoint collection such that for each $F \in \mathcal{F}$ we have $E_{\mathcal{F}}(F) \subseteq F$ and $\sigma(F) \le 2\sigma(E_{\mathcal{F}}(F))$. Then*

$$(\sum_{F\in\mathcal{F}} (\langle |f|\rangle_F^\sigma)^p \sigma(F))^{1/p} \le 2^{1/p} p' \|f\|_{L^p(\sigma)}.$$

Proof of the lemma. By the definition of the Hardy-Littlewood maximal function we have $\langle |f|\rangle_F^\sigma \le \inf_F M^\sigma f$. Moreover we have the norm inequality $\|M^\sigma f\|_{L^p(\sigma)} \le p'\|f\|_{L^p(\sigma)}$. These facts together with the assumptions yield

$$(\sum_{F\in\mathcal{F}} (\langle |f|\rangle_F^\sigma)^p \sigma(F))^{1/p} \le 2^{1/p} (\sum_{F\in\mathcal{F}} \int_{E_{\mathcal{F}}(F)} (\inf_F M^\sigma f)\sigma)^{1/p} \le 2^{1/p} \|M^\sigma f\|_{L^p(\sigma)} \le 2^{1/p} p' \|f\|_{L^p(\sigma)}.$$

□

2.4. **Applying the dual testing condition.** Let us first consider the summation (2.7a). Assume for the moment that we may replace f in the summation (2.7a) with functions f_G that satisfy

$$(\sum_{G\in\mathcal{G}} \|f_G\|_{L^p(\sigma)}^p)^{1/p} \le 5p'\|f\|_{L^p(\sigma)}. \tag{2.8}$$

Then we have

$$\begin{aligned}
&\sum_{G\in\mathcal{G}} \sum_{\substack{F\in\mathcal{F}\\ F\subseteq G}} \sum_{\substack{Q\in\mathcal{D}\\ \pi(Q)=(F,G)}} \lambda_Q \langle g\omega\rangle_Q \int_Q f_G \sigma \\
&\le \sum_{G\in\mathcal{G}} \int (\sum_{\substack{Q\in\mathcal{D}:\\ \pi_{\mathcal{G}}(Q)=G}} \lambda_Q \langle g_Q \omega\rangle_Q 1_Q) f_G\sigma && \text{by relaxing the summation condition} \\
&= \sum_{G\in\mathcal{G}} \int (\sum_{Q\in\mathcal{D}} \lambda_Q \langle (g_G)_Q \omega\rangle_Q 1_Q) f_G\sigma && \text{by defining } g_G := (g_Q)_{\substack{Q\in\mathcal{D}:\\ \pi_{\mathcal{G}}(Q)=G}} \\
&= \sum_{G\in\mathcal{G}} \langle f_G, T_G^*(g_G\,\omega)\rangle_{L^p(\sigma)\times L^{p'}(\sigma)} \\
&\le \sum_{G\in\mathcal{G}} \|f_G\|_{L^p(\sigma)} \|T_G^*(g_G\,\omega)\|_{L^{p'}(\sigma)} && \text{by Hölder's inequality} \\
&\le C^* \sum_{G\in\mathcal{G}} \|f_G\|_{L^p(\sigma)} \|g_G\|_{L^\infty_{\ell^{r'}}(\omega)}\, \omega(G)^{1/p'} && \text{by the testing condition (1.2b)} \\
&\le 2C^* \sum_{G\in\mathcal{G}} \|f_G\|_{L^p(\sigma)} \langle |g|_{r'}\rangle_G^\omega \omega(G)^{1/p'} && \text{by the property (b5)} \\
&\le 2C^* (\sum_{G\in\mathcal{G}} \|f_G\|_{L^p(\sigma)}^p)^{1/p} (\sum_{G\in\mathcal{G}} (\langle |g|_{r'}\rangle_G^\omega)^{p'} \omega(G))^{1/p'} && \text{by Hölder's inequality} \\
&\le 2^{1+1/p'} p C^* (\sum_{G\in\mathcal{G}} \|f_G\|_{L^p(\sigma)}^p)^{1/p} \|g\|_{L^{p'}_{\ell^{r'}}(\omega)} && \text{by Lemma 2.1} \\
&\le 4pC^* 5p' \|f\|_{L^p(\sigma)} \|g\|_{L^{p'}_{\ell^{r'}}(\omega)} && \text{by the claimed inequality (2.8).}
\end{aligned}$$

Next we prove that we may replace f in the summation (2.7a) with functions f_G that satisfy the claimed inequality (2.8).

Claim. In the summation (2.7a) we may replace f with functions f_G that satisfy

$$\Big(\sum_{G\in\mathcal{G}} \|f_G\|_{L^p(\sigma)}^p\Big)^{1/p} \le 5p'\|f\|_{L^p(\sigma)}.$$

Proof of the claim. Since the summation condition $\pi_{\mathcal{G}}(Q) = G$ implies that $Q \subseteq G$ and since the sets $G' \in \mathrm{ch}_{\mathcal{G}}(G)$ and $E_{\mathcal{G}}(G)$ partition G, we have

$$\int_Q f\sigma = \int_{Q\cap E_{\mathcal{G}}(G)} f\sigma + \sum_{G'\in \mathrm{ch}_{\mathcal{G}}(G)} \int_{Q\cap G'} f\sigma.$$

We may suppose that $Q\cap G' \ne \varnothing$ because otherwise the integral over $Q\cap G'$ vanishes. Then either $G' \subsetneq Q$ or $Q \subseteq G'$, the latter which is excluded by the summation condition $\pi_{\mathcal{G}}(Q) = G$. Hence we may restrict the summation index set $\{G' \in \mathrm{ch}_{\mathcal{G}}(G)\}$ to the set $\{G' \in \mathrm{ch}_{\mathcal{G}}(G) : G' \subsetneq Q\}$. Therefore

$$\int_{Q\cap G'} f\sigma = \int_{G'} f\sigma = \langle f\rangle_{G'}^{\sigma}\sigma(G') = \int_Q \langle f\rangle_{G'}^{\sigma} 1_{G'}\sigma.$$

The summation conditions $\pi_{\mathcal{F}}(Q) = F$ and $F \subseteq G$ imply that $Q \subseteq F \subseteq G$. Therefore

$$\begin{aligned}\{G' \in \mathrm{ch}_{\mathcal{G}}(G) : G' \subsetneq Q\} &\subseteq \{G' \in \mathrm{ch}_{\mathcal{G}}(G) : G' \subseteq F \subseteq G \text{ for some } F \in \mathcal{F}\}\\ &= \bigcup_{\substack{F\in\mathcal{F}:\\ \pi_{\mathcal{G}}(F)=G}} \{G' \in \mathrm{ch}_{\mathcal{G}}(G) : \pi_{\mathcal{F}}(G') = F\} =: \mathrm{ch}_{\mathcal{G}}^*(G).\end{aligned} \tag{2.9}$$

Altogether we have

$$\int_Q f\sigma \le \int_Q \Big(f 1_{E_{\mathcal{G}}(G)} + \sum_{G'\in \mathrm{ch}_{\mathcal{G}}^*(G)} \langle f\rangle_{G'}^{\sigma} 1_{G'}\Big)\sigma =: \int_Q f_G\sigma.$$

Next we check the claimed inequality (2.8). By the triangle inequality we have

$$\|f_G\|_{L^p(\sigma)} \le \|f 1_{E_{\mathcal{G}}(G)}\|_{L^p(\sigma)} + \Big\|\sum_{G'\in \mathrm{ch}_{\mathcal{G}}^*(G)} \langle f\rangle_{G'}^{\sigma} 1_{G'}\Big\|_{L^p(\sigma)},$$

which by the triangle inequality and by the pairwise disjointness of each of the collections $\{G'\}_{G'\in \mathrm{ch}_{\mathcal{G}}^*(G)}$ and $\{E_{\mathcal{G}}(G)\}_{G\in\mathcal{G}}$ implies that

$$\begin{aligned}\Big(\sum_{G\in\mathcal{G}} \|f_G\|_{L^p(\sigma)}^p\Big)^{1/p} &\le \Big(\sum_{G\in\mathcal{G}} \|f 1_{E_{\mathcal{G}}(G)}\|_{L^p(\sigma)}^p\Big)^{1/p} + \Big(\sum_{G\in\mathcal{G}} \Big\|\sum_{G'\in \mathrm{ch}_{\mathcal{G}}^*(G)} \langle f\rangle_{G'}^{\sigma} 1_{G'}\Big\|_{L^p(\sigma)}^p\Big)^{1/p}\\ &\le \Big(\Big\|\sum_{G\in\mathcal{G}} f 1_{E_{\mathcal{G}}(G)}\Big\|_{L^p(\sigma)}^p\Big)^{1/p} + \Big(\sum_{G\in\mathcal{G}} \sum_{G'\in \mathrm{ch}_{\mathcal{G}}^*(G)} \|\langle f\rangle_{G'}^{\sigma} 1_{G'}\|_{L^p(\sigma)}^p\Big)^{1/p}\\ &\le \|f\|_{L^p(\sigma)} + \Big(\sum_{G\in\mathcal{G}} \sum_{G'\in \mathrm{ch}_{\mathcal{G}}^*(G)} (\langle f\rangle_{G'}^{\sigma})^p \sigma(G')\Big)^{1/p}.\end{aligned}$$

We can estimate the last term as follows.

$$
\begin{aligned}
&\sum_{G\in\mathcal{G}}\sum_{G'\in\mathrm{ch}^*_{\mathcal{G}}(G)}(\langle f\rangle^\sigma_{G'})^p\sigma(G') \\
&=\sum_{G\in\mathcal{G}}\sum_{\substack{F\in\mathcal{F}:\\ \pi_{\mathcal{G}}(F)=G}}\sum_{\substack{G'\in\mathrm{ch}_{\mathcal{G}}(G):\\ \pi_{\mathcal{F}}(G')=F}}(\langle f\rangle^\sigma_{G'})^p\sigma(G') && \text{by (2.9)}\\
&\le\sum_{G\in\mathcal{G}}\sum_{\substack{F\in\mathcal{F}:\\ \pi_{\mathcal{G}}(F)=G}}2^p(\langle f\rangle^\sigma_{F})^p\Big(\sum_{\substack{G'\in\mathrm{ch}_{\mathcal{G}}(G):\\ \pi_{\mathcal{F}}(G')=F}}\sigma(G')\Big) && \text{by the property (a4)}\\
&\le 2^p\sum_{G\in\mathcal{G}}\sum_{\substack{F\in\mathcal{F}:\\ \pi_{\mathcal{G}}(F)=G}}(\langle f\rangle^\sigma_{F})^p\sigma(F) && \text{because } \mathrm{ch}_{\mathcal{G}}(G) \text{ is pairwise disjoint}\\
&=2^p\sum_{F\in\mathcal{F}}(\langle f\rangle^\sigma_{F})^p\sigma(F) && \text{because } \mathcal{F}=\bigcup_{G\in\mathcal{G}}\{F\in\mathcal{F}:\pi_{\mathcal{G}}(F)=G\}\\
&\le 2^{p+1}(p')^p\|f\|^p_{L^p(\sigma)} && \text{by Lemma 2.1.}
\end{aligned}
$$

Altogether

$$
\Big(\sum_{G\in\mathcal{G}}\|f_G\|^p_{L^p(\sigma)}\Big)^{1/p}\le\|f\|_{L^p(\sigma)}+2^{1/p+1}p'\|f\|_{L^p(\sigma)}\le 5p'\|f\|_{L^p(\sigma)}.
$$

This concludes the proof of the claim. □

2.5. **Applying the direct testing condition.** Next we estimate the summation (2.7b).

$$
\begin{aligned}
&\sum_{F\in\mathcal{F}}\sum_{\substack{G\in\mathcal{G}\\ G\subseteq F}}\sum_{\substack{Q\in\mathcal{D}\\ \pi(Q)=(F,G)}}\lambda_Q\langle f\rangle^\sigma_Q\langle\sigma\rangle_Q\int_Q g_Q\omega \\
&\le 2\sum_{F\in\mathcal{F}}\langle f\rangle^\sigma_F\sum_{\substack{G\in\mathcal{G}\\ G\subseteq F}}\sum_{\substack{Q\in\mathcal{D}\\ \pi(Q)=(F,G)}}\lambda_Q\langle\sigma\rangle_Q\int_Q g_Q\omega && \text{by the property (a4)}\\
&=2\sum_{F\in\mathcal{F}}\langle f\rangle^\sigma_F\int\sum_{Q\in\mathcal{D}}(T_F(\sigma))_Q(g_F)_Q\omega && \text{by } g_F:=(g_Q)_{\substack{Q\in\mathcal{D}:\\ \pi(Q)=(F,G)\text{ for some}\\ G\in\mathcal{G}\text{ such that }G\subseteq F}}\\
&=2\sum_{F\in\mathcal{F}}\langle f\rangle^\sigma_F\langle T_F(\sigma),g_F\rangle_{L^p_{\ell^r}(\omega)\times L^{p'}_{\ell^{r'}}(\omega)}\\
&\le 2\sum_{F\in\mathcal{F}}\langle f\rangle^\sigma_F\|T_F(\sigma)\|_{L^p_{\ell^r}(\omega)}\|g_F\|_{L^{p'}_{\ell^{r'}}(\omega)} && \text{by Hölder's inequality}\\
&\le 2C\sum_{F\in\mathcal{F}}\langle f\rangle^\sigma_F\sigma(F)^{1/p}\|g_F\|_{L^{p'}_{\ell^{r'}}(\omega)} && \text{by the testing condition (1.2a)}\\
&\le 2C\Big(\sum_{F\in\mathcal{F}}(\langle f\rangle^\sigma_F)^p\sigma(F)\Big)^{1/p}\Big(\sum_{F\in\mathcal{F}}\|g_F\|^{p'}_{L^{p'}_{\ell^{r'}}(\omega)}\Big)^{1/p'} && \text{by Hölder's inequality}\\
&\le 2C2p'\|f\|_{L^p(\sigma)}\Big(\sum_{F\in\mathcal{F}}\|g_F\|^{p'}_{L^{p'}_{\ell^{r'}}(\omega)}\Big)^{1/p'} && \text{by Lemma 2.1.}
\end{aligned}
$$

The proof of the following claim completes the proof of the theorem.

Claim. We have

$$
\Big(\sum_{F\in\mathcal{F}}\|g_F\|^{p'}_{L^{p'}_{\ell^{r'}}(\omega)}\Big)^{1/p'}\le 5p\|g\|_{L^{p'}_{\ell^{r'}}(\omega)}. \tag{2.10}
$$

Proof of the claim. By definition the components of the function $g_F = (a_Q 1_Q)_{Q\in\mathcal{I}(F)}$ are indexed by the set

$$\mathcal{I}(F) = \{Q \in \mathcal{D} : \pi(Q) = (F,G) \text{ for some } G \in \mathcal{G} \text{ such that } G \subseteq F\}.$$

The function g_F is supported on $\bigcup_{Q\in\mathcal{I}(F)} Q$. Since the condition $\pi_{\mathcal{F}}(Q) = F$ implies that $Q \subseteq F$, we have that $\bigcup_{Q\in\mathcal{I}(F)} Q \subseteq F$. Since the sets $F' \in \mathrm{ch}_{\mathcal{F}}(F)$ and $E_{\mathcal{F}}(F)$ partition F, we have

$$g_F = g_F 1_{E_{\mathcal{F}}(F)} + \sum_{F'\in\mathrm{ch}_{\mathcal{F}}(F)} g_F 1_{F'}.$$

By the triangle inequality we have

$$\|g_F\|_{L^{p'}_{\ell^{r'}}(\omega)} \le \|g_F 1_{E_{\mathcal{F}}(F)}\|_{L^{p'}_{\ell^{r'}}(\omega)} + \|\sum_{F'\in\mathrm{ch}_{\mathcal{F}}(F)} g_F 1_{F'}\|_{L^{p'}_{\ell^{r'}}(\omega)},$$

which by the triangle inequality, by the fact $|g_F|_{r'} \le |g|_{r'}$ and by the pairwise disjointness of each of the collections $\{E_{\mathcal{F}}(F)\}_{F\in\mathcal{F}}$ and $\{F'\}_{F'\in\mathrm{ch}_{\mathcal{F}}(F)}$ implies that

$$\begin{aligned}
\Big(\sum_{F\in\mathcal{F}} \|g_F\|^{p'}_{L^{p'}_{\ell^{r'}}(\omega)}\Big)^{1/p'} &\le \Big(\sum_{F\in\mathcal{F}} \|g_F 1_{E_{\mathcal{F}}(F)}\|^{p'}_{L^{p'}_{\ell^{r'}}(\omega)}\Big)^{1/p'} + \Big(\sum_{F\in\mathcal{F}} \|\sum_{F'\in\mathrm{ch}_{\mathcal{F}}(F)} g_F 1_{F'}\|^{p'}_{L^{p'}_{\ell^{r'}}(\omega)}\Big)^{1/p'} \\
&\le \Big(\|\sum_{F\in\mathcal{F}} g 1_{E_{\mathcal{F}}(F)}\|^{p'}_{L^{p'}_{\ell^{r'}}(\omega)}\Big)^{1/p'} + \Big(\sum_{F\in\mathcal{F}} \sum_{F'\in\mathrm{ch}_{\mathcal{F}}(F)} \|g_F 1_{F'}\|^{p'}_{L^{p'}_{\ell^{r'}}(\omega)}\Big)^{1/p'} \\
&\le \|g\|_{L^{p'}_{\ell^{r'}}(\omega)} + \Big(\sum_{F\in\mathcal{F}} \sum_{F'\in\mathrm{ch}_{\mathcal{F}}(F)} \|g_F 1_{F'}\|^{p'}_{L^{p'}_{\ell^{r'}}(\omega)}\Big)^{1/p'}.
\end{aligned}$$

It remains to estimate the last term. Consider the integral

$$\|g_F 1_{F'}\|^{p'}_{L^{p'}_{\ell^{r'}}(\omega)} = \int_{F'} |g_F|^{p'}_{r'} = \begin{cases} \int_{F'} \big((\sum_{Q\in\mathcal{I}(F)} a_Q^{r'} 1_Q(x))^{1/r'}\big)^{p'} \omega(x)\mathrm{d}x & \text{if } 1 \le r' < \infty, \\ \int_{F'} \big(\sup_{Q\in\mathcal{I}(F)} (a_Q 1_Q(x))\big)^{p'} \omega(x)\mathrm{d}x & \text{if } r' = \infty. \end{cases}$$

Let $Q \in \mathcal{I}(F)$ and $F' \in \mathrm{ch}_{\mathcal{F}}(F)$. The cubes Q and F' for which $Q \cap F' = \varnothing$ do not contribute to the integral. Hence we may restrict to the cubes such that $Q \cap F' \ne \varnothing$. Then by nestedness either $F' \subsetneq Q$ or $Q \subseteq F'$, the latter which is excluded by the condition $\pi_{\mathcal{F}}(Q) = F$. Hence $F' \subsetneq Q$. Moreover, we have that $\pi_{\mathcal{G}}(Q) = G$ for some $G \subseteq F$, which implies that $Q \subseteq G \subseteq F$. Altogether we have $F' \subsetneq Q \subseteq G \subseteq F$. Therefore we may replace the summation over the index set $\mathrm{ch}_{\mathcal{F}}(F)$ with the summation over the set

$$\begin{aligned}
\mathrm{ch}^*_{\mathcal{F}}(F) &= \{F' \in \mathrm{ch}_{\mathcal{F}}(F) : F' \subseteq G \subseteq F \text{ for some } G \in \mathcal{G}\} \\
&= \bigcup_{\substack{G\in\mathcal{G}: \\ \pi_{\mathcal{F}}(G)=F}} \{F' \in \mathrm{ch}_{\mathcal{F}}(F) : \pi_{\mathcal{G}}(F') = G\}.
\end{aligned} \tag{2.11}$$

and we may replace the index set $\mathcal{I}(F)$ with the index set

$$\mathcal{I}(F,F') := \{Q \in \mathcal{D} : Q \supsetneq F' \text{ and } \pi(Q) = (F,G) \text{ for some } G \in \mathcal{G} \text{ such that } G \subseteq F\}. \tag{2.12}$$

By the containment

$$\mathcal{I}(F,F') \subseteq \{Q \in \mathcal{D} : Q \supsetneq F'\}$$

and the property (b4) we have

$$|(a_Q)_{Q\in\mathcal{I}(F,F')}|_{r'} \le |(a_Q)_{\substack{Q\in\mathcal{D}: \\ Q\supsetneq F'}}|_{r'} \le 2\langle |g|_{r'} \rangle^{\omega}_{\pi_{\mathcal{G}}(F')}. \tag{2.13}$$

Therefore

$$
\begin{aligned}
&\sum_{F\in\mathcal{F}}\sum_{F'\in\mathrm{ch}_{\mathcal{F}}(F)}\|g_F 1_{F'}\|^{p'}_{L^{p'}_{\ell^{r'}}(\omega)} \\
&= \sum_{F\in\mathcal{F}}\sum_{F'\in\mathrm{ch}^*_{\mathcal{F}}(F)}\|(a_Q)_{Q\in\mathcal{I}(F,F')}1_{F'}\|^{p'}_{L^{p'}_{\ell^{r'}}(\omega)} && \text{the replacements (2.11) and (2.12)}\\
&\le \sum_{F\in\mathcal{F}}\sum_{F'\in\mathrm{ch}^*_{\mathcal{F}}(F)} 2^{p'}(\langle|g|_{r'}\rangle^{\omega}_{\pi_{\mathcal{G}}(F')})^{p'}\omega(F') && \text{by (2.13)}\\
&\le \sum_{F\in\mathcal{F}}\sum_{\substack{G\in\mathcal{G}\\ \pi_{\mathcal{F}}(G)=F}}\sum_{\substack{F'\in\mathrm{ch}_{\mathcal{F}}(F)\\ \pi_{\mathcal{G}}(F')=G}} 2^{p'}(\langle|g|_{r'}\rangle^{\omega}_{\pi_{\mathcal{G}}(F')})^{p'}\omega(F') && \text{by (2.11)}\\
&= 2^{p'}\sum_{F\in\mathcal{F}}\sum_{\substack{G\in\mathcal{G}\\ \pi_{\mathcal{F}}(G)=F}}(\langle|g|_{r'}\rangle^{\omega}_{G})^{p'}\Big(\sum_{\substack{F'\in\mathrm{ch}_{\mathcal{F}}(F)\\ \pi_{\mathcal{G}}(F')=G}}\omega(F')\Big)\\
&\le 2^{p'}\sum_{F\in\mathcal{F}}\sum_{\substack{G\in\mathcal{G}\\ \pi_{\mathcal{F}}(G)=F}}(\langle|g|_{r'}\rangle^{\omega}_{G})^{p'}\omega(G) && \text{because } \mathrm{ch}_{\mathcal{F}}(F) \text{ is pairwise disjoint}\\
&= 2^{p'}\sum_{G\in\mathcal{G}}(\langle|g|_{r'}\rangle^{\omega}_{G})^{p'}\omega(G) && \text{because } \mathcal{G}=\bigcup_{F\in\mathcal{F}}\{G\in\mathcal{G}:\pi_{\mathcal{F}}(G)=F\}\\
&\le 2^{p'+1}p^{p'}\|g\|^{p'}_{L^{p'}_{\ell^{r'}}(\omega)} && \text{by Lemma 2.1.}
\end{aligned}
$$

Altogether

$$
\Big(\sum_{F\in\mathcal{F}}\|g_F\|^{p'}_{L^{p'}_{\ell^{r'}}(\omega)}\Big)^{1/p'} \le \|g\|_{L^{p'}_{\ell^{r'}}(\omega)} + 2^{1/p'+1}p\|g\|_{L^{p'}_{\ell^{r'}}(\omega)} \le 5p\|g\|_{L^{p'}_{\ell^{r'}}(\omega)}.
$$

This completes the proof of the claim. □

□

Remark 2.3. In fact each of the proofs [3, 8, 1] for $r=1$, the proof [7] for $1<r<\infty$, and the proof [5] for $r=\infty$ each works in the case $T(\cdot\omega):L^p(\sigma)\to L^q_{\ell^r}(\omega)$ with $1<p\le q<\infty$. Also the proof of this note works in that case by using the following facts. For $p'\ge q'$ the estimate $\|\cdot\|_{\ell^{p'}}\le\|\cdot\|_{\ell^{q'}}$ implies that

$$
\Big(\sum_{G\in\mathcal{G}}(\langle|g|_{r\prime}\rangle^{\omega}_{G})^{p'}\omega(G)\Big)^{1/p'} \le \Big(\sum_{G\in\mathcal{G}}(\langle|g|_{r\prime}\rangle^{\omega}_{G})^{q'}\omega(G)\Big)^{1/q'}
$$

and that

$$
\Big(\sum_{F\in\mathcal{F}}\|g_F\|^{p'}_{L^{q'}_{\ell^{r'}}(\omega)}\Big)^{1/p'} \le \Big(\sum_{F\in\mathcal{F}}\|g_F\|^{q'}_{L^{q'}_{\ell^{r'}}(\omega)}\Big)^{1/q'}.
$$

Moreover, the estimate (2.10) holds for every p', hence in particular for q', as it is seen from the proof of the estimate.

Acknowledgment. The author would like to thank Tuomas Hytönen for suggesting the author to study whether Hytönen's proof [1, Section 6] of the case $r=1$ could be adapted to the case $1<r<\infty$, for pointing out how to simplify certain details of the proof, and for encouraging and insightful discussions.

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